Benha University
Faculty of Engineering- Shoubra
Mechanical Engineering Department
2nd Year Mechanical (production)

Final Term Exam
Date:14th of January 2012
Mathematics 2
Duration : 3 hours

- Answer all the following questions
- Illustrate your answers with sketches when necessary.
- The exam. Consists of one page
- No. of questions:3
- Total Mark: 80 Marks

1a) Evaluate the following integrals

i) $\int_0^{\infty} \frac{t^{c+1}}{(1+t^2)^2} dt$, ii) $\int_0^1 \frac{t^{3c-m}}{\sqrt[3]{(1-t^3)}} dt$, iii) $\int_{-\pi/6}^{\pi/3} (\sqrt{3}\sin\theta + \cos\theta)^{1/4} d\theta$, iv) $\int_0^{\infty} a^{-mx^n} dx$,

v) $\int_0^{\infty} t^2 \cos 3t\, e^{-5t} dt$, vi) $\int_0^1 x^m (\log_a x)^n dx$, $\int_c \frac{z^3+5z+7}{(z-i)^2} dz$ c: $|z-2| + |z+2| = 6$

(21 marks)

1b) Find Laplace Transform for the following functions

i) $f(t) = \frac{e^{2t} - e^{-3t}}{t^2} + \cos 2t \cosh 5t$, ii) $g(t) = \begin{cases} t^2 + 1 & t < 1 \\ e^{-3t} + 1 & 1 < t < 2 \\ 1 & t > 2 \end{cases} + t \int_{u=0}^{t} e^{-2u} \sin u\, du$

(8 marks)

2a) Find inverse Laplace for the following functions:

i) $F(S) = \frac{e^{-2\pi s}}{(s+1)^2 + 4} + \frac{25}{s^3(s^2+4s+4)}$, ii) $G(S) = \frac{9s+4}{(s+3)^3} + \frac{s+5}{s^2+6s+20}$

(8 marks)

2b-i) If $f(z) = e^x\cos(ay) + i\, e^x\sin(y-b)$ is differentiable at every point, then find a and b.

(5 marks)

2b-ii) Find residues of $f(z) = \frac{z^3+5z+7}{(z-1)^2(z^2-3z-4)}$ using Residue theorem – Laurant series.

(5 marks)

2b-iii) Expand in Fourier series

I) $f(x) = \begin{cases} \pi/2 + x, & -\pi \le x \le 0 \\ \pi/2 - x, & 0 < x \le \pi \end{cases}$, then deduce the sum $\frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$

II) $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ -(x-\pi), & \pi < x \le 2\pi \end{cases}$, then find $\sum_{n=1}^{\infty} \frac{1}{n}$

(10 marks)

3-a) Solve using Laplace Transform the following differential equations:

i) $y' - 2y = 8t$, $y(0) = 3$,

ii) $y'' + y = -12 e^{-2t} \cos t$, $y(\pi/4) = \pi/2$, $y'(\pi/4) = 2 - \sqrt{2}$

(8 marks)

3b-i) Find the constants of the curve y(x) = acosx + bsinx that fit the data: (-1,2), (3,4), (6,9)

(7 marks)

3b-ii) Solve the following system of equations using Picard up to 2nd approximation

$$x' - 3y' = -2t + x - 2y - 7,\quad 2x' + y' = 10t + y + 3 - t^2,\quad x(0) = 1,\ y(0) = -3$$

(8 marks)

Good Luck BOARD OF EXAMINERS: **Dr. Khaled El Naggar**

Model answer

1a-i) Put $y = t^2$, therefore $dt = \frac{1}{2} y^{-\frac{1}{2}} dy$, therefore $\int_0^\infty \frac{t^{c+1}}{(1+t^2)^2} dt = \frac{1}{2}\int_0^\infty \frac{y^{c/2}}{(1+y)^2} dy = \frac{1}{2}\beta(m,n)$,

where m-1 = c/2, m+n = 2

1a-ii) Put $y = t^3$, therefore $dt = \frac{1}{3} y^{-\frac{2}{3}} dy$, therefore $\int_0^1 \frac{t^{3c-m}}{\sqrt[3]{(1-t^3)}} dt = \frac{1}{3}\int_0^1 \frac{y^{(3c-m)/3} y^{-2/3}}{\sqrt[3]{(1-y)}} dy$

1a-iii) $\int_{-\pi/6}^{\pi/3} (\sqrt{3}\sin\theta + \cos\theta)^{1/4} d\theta$

$= 2\int_{-\pi/6}^{\pi/3} (\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta)^{1/4} d\theta = 2\int_{-\pi/6}^{\pi/3} [\cos(\pi/6)\sin\theta + \cos(\pi/6)\cos\theta]^{1/4} d\theta$, put $y = \theta + \pi/6$, thus

$\int_{-\pi/6}^{\pi/3} (\sqrt{3}\sin\theta + \cos\theta)^{1/4} d\theta = \int_{-\pi/6}^{\pi/3} [\sin(\theta + \pi/6)]^{1/4} d\theta = \int_0^{\pi/2} [\sin y]^{1/4} dy = \frac{1}{2}\beta(m,n)$, 2m-1 = ¼, 2n-1 =0

1a-vii) Since z = i is inside contour, therefore $\int_c \frac{z^3 + 5z + 7}{(z-i)^2} dz = 2\pi i \; f`(i) = 4\pi i$

1b-i) $F(s) = \int_s^\infty [\ln(s+3) - \ln(s-2)]\, ds + \frac{1}{2}[\frac{s-5}{(s-5)^2+4} + \frac{s+5}{(s+5)^2+4}]$

1b-ii) $G(s) = \int_0^1 [t^2 + 1]\, e^{-st}\, dt + \int_1^2 [e^{-3t} + 1]\, e^{-st}\, dt + \int_2^\infty e^{-st}\, dt - \frac{d}{ds}[\frac{1}{s[(s+2)^2+1]}]$

2a-i)Since $L^{-1}\{ \frac{25}{s(s+2)^2} \} = 25 \int_{u=0}^{t} u\, e^{-2u}\, du = \frac{-25}{4}[(2t+1)e^{-2t} - 1]$, therefore $L^{-1}\{ \frac{1}{s^2(s+2)^2} \} =$

$(-25/4) \int_{u=0}^{t} [(2u+1)e^{-2u} - 1]\, du = (25/4)[(t+1)\, e^{-2t} + t - 1]$, therefore $L^{-1}\{ \frac{1}{s^3(s+2)^2} \} =$

$(25/4) \int_{u=0}^{t} [(u+1)e^{-2u} + u - 1]\, du$ and $L^{-1}\{ \frac{e^{-2\pi s}}{(s+1)^2+4} \} = (½) \sin 2(t-2\pi)\, e^{-(t-2\pi)}\, u(t-2\pi)$

2a-ii) Since $\frac{9s+4}{(s+3)^3} = \frac{9(s+3)-23}{(s+3)^3}$, therefore $L^{-1}\{ \frac{9s+4}{(s+3)^3}) = 9t\, e^{-3t} - (23/2)\, t^2 e^{-3t}$ and $\frac{s+5}{s^2+6s+20} =$

$\frac{(s+3)+2}{(s+3)^2+11}$, therefore $L^{-1}\{ \frac{s+5}{s^2+6s+20} \} = e^{-3t}\, [\cos\sqrt{11}\, t + (2/\sqrt{11}) \sin\sqrt{11}\, t]$

2b-i) Since the function is differentiable, therefore f(z) is analytic, i.e. Cauchy-Riemann equation is satisfied, hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, thus $e^x\cos(ay) = e^x\cos(y-b)$, then a= 1, b = 0.

2b-ii) $f(z) = \frac{z^3+5z+7}{(z-1)^2(z^2-3z-4)} = \frac{z^3+5z+7}{(z-1)^2(z-4)(z+1)}$, $\text{Res}_{z=1} = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2\frac{z^3+5z+7}{(z-1)^2(z^2-3z-4)}$

$= \lim_{z\to 1} \frac{(3z^2+5)(z^2-3z-4) - (2z-3)(z^3+5z+7)}{(z^2-3z-4)^2} = \frac{-35}{36}$, $\text{Res}_{z=-1} = \lim_{z\to -1} (z+1)\frac{z^3+5z+7}{(z-1)^2(z-4)(z+1)} =$

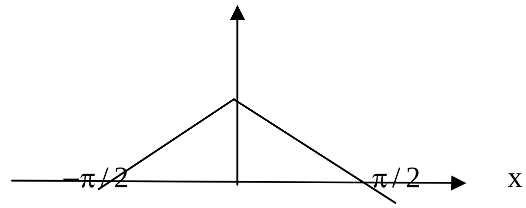
$\frac{-1}{20}$, $\text{Res}_{z=4} = \lim_{z\to 4} (z-4)\frac{z^3+5z+7}{(z-1)^2(z-4)(z+1)} = \frac{91}{45}$.

Using Laurant series $\dfrac{z^3+5z+7}{(z-1)^2(z^2-3z-4)} = \dfrac{121z-199}{36(z-1)^2} + \dfrac{91}{45(z-4)} - \dfrac{1}{20(z+1)} =$

$= \dfrac{-35z-43}{36z^2(1-\frac{1}{z})^2} + \dfrac{91}{45z(1-\frac{4}{z})} - \dfrac{1}{20z(1+\frac{1}{z})}$, where $|z|>4$

2b-iii)

I) The function is even, therefore



$a_0 = \dfrac{2}{\pi}\int_0^{\pi}[(\pi/2)-x]\,dx = 0$,

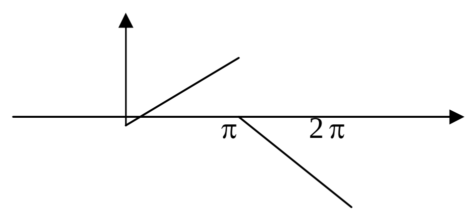
$a_n = \dfrac{2}{\pi}\int_0^{\pi}[(\pi/2)-x]\cos(nx)\,dx = \dfrac{2}{\pi}[[(\pi/2)-x](\dfrac{1}{n}\sin(nx)) + (\dfrac{-1}{n^2}\cos(nx))]_0^{\pi} = \dfrac{2}{n^2\pi}(1-\cos n\pi)$

Therefore $a_{2n}=0$ and $a_{2n-1} = \dfrac{4}{(2n-1)^2\pi}$, thus $f(x) = \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\cos(2n-1)x$

By using parseval, we get $\dfrac{1}{\pi}[\int_{-\pi}^{0}[(\pi/2)+x]^2\,dx + \int_0^{\pi}[(\pi/2)-x]^2\,dx] = \dfrac{16}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^4} = \dfrac{\pi^2}{6}$,

therefore $\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^4} = \dfrac{\pi^4}{96}$, thus $\dfrac{1}{1^4}+\dfrac{1}{3^4}+\dfrac{1}{5^4}+\dfrac{1}{7^4}+.... = \dfrac{\pi^4}{96}$, hence $\dfrac{1}{3^4}+\dfrac{1}{5^4}+\dfrac{1}{7^4}+.... = \dfrac{\pi^4}{96}-1$

II) The function is odd harmonic, therefore



$a_{2n-1} = \dfrac{2}{\pi}\int_0^{\pi} x\cos(2n-1)x\,dx = \dfrac{2}{\pi}(x(\dfrac{\sin(2n-1)x}{2n-1}) - (\dfrac{-\cos(2n-1)x}{(2n-1)^2}))_0^{\pi} = -\dfrac{4}{\pi(2n-1)^2}$

$b_{2n-1} = \dfrac{2}{\pi}\int_0^{\pi} x\ \sin(2n-1)x\,dx = \dfrac{2}{\pi}(x(\dfrac{-\cos(2n-1)x}{2n-1}) - (\dfrac{-\sin(2n-1)x}{(2n-1)^2}))_0^{\pi} = \dfrac{2}{(2n-1)}$

Thus $f(x) = \sum_{n=1}^{\infty} a_{2n-1}\cos(2n-1)x + \sum_{n=1}^{\infty} b_{2n-1}\sin(2n-1)x$

Put x = 0, therefore $\sum_{n=1}^{\infty} -\dfrac{4}{\pi(2n-1)^2} = 0$, therefore $\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2} = 0$

3a-i) By taking Laplace to both sides, we get $Y(s) = \dfrac{8}{s^2(s-2)} + \dfrac{3}{(s-2)}$, therefore y(t) =

$4\int_{u=0}^{t}[e^{2u}-1]\,du + 3e^{2t}$

3a-ii) By taking Laplace to both sides, we get s^2 Y(s) –sy(0) –y`(0) + Y(s) = $\frac{-12(s+2)}{(s+2)^2+1}$

Therefore Y(s) = $\frac{-12(s+2)}{[(s+2)^2+1][s^2+1]} + \frac{as}{[s^2+1]} + \frac{b}{[s^2+1]}$ by assuming y(0)=a and y`(0) = b

Thus Y(s) = $\frac{As+B}{[(s+2)^2+1]} + \frac{Cs+D}{[s^2+1]} + \frac{as}{[s^2+1]} + \frac{b}{[s^2+1]}$, therefore

y(t) = A e^{-2t} cost +(B-2A) e^{-2t} sint + Ccost +Dsint +acost + bsint,

but y (π/4) = π/2 , y` (π/4) = 2-$\sqrt{2}$, therefore wecan get a and b.

3b-i) If we consider the function y = a $\phi_0(x)$ + b $\phi_1(x)$ such that

$$\sum_{i=1}^{N} y_i\phi_0(x_i) = a\sum_{i=1}^{N}\phi_0^2(x_i) + b\sum_{i=1}^{N}\phi_0(x_i)\phi_1(x_i), \quad \sum_{i=1}^{N} y_i\phi_1(x_i) = a\sum_{i=1}^{N}\phi_0(x_i)\phi_1(x_i) + b\sum_{i=1}^{N}\phi_1^2(x_i),$$

where $\phi_0(x) = \cos(x)$ and $\phi_1(x) = \sin x$, $\sum_{i=1}^{3}\phi_0(x_i)\phi_1(x_i) = -0.86$, $\sum_{i=1}^{3}\phi_0^2(x_i) = 2.19$, $\sum_{i=1}^{3}\phi_1^2(x_i) = 0.81$,

$\sum_{i=1}^{3}\phi_0(x_i)y_i = 5.76$, $\sum_{i=1}^{3}\phi_1(x_i)y_i = -3.63$, therefore: 5.76 = 2.19a -0.86b, -3.63 = -0.86a + 0.81b